Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 12, 2003*
Accepted: *October 3, 2003*

Topological charges for branes in M-theory

Emily J. Hackett-Jones and Douglas J. Smith

Centre for Particle Theory, Department of Mathematical Sciences
University of Durham
Durham DH1 3LE, UK
E-mail: e.j.hackett-jones@durham.ac.uk, douglas.smith@durham.ac.uk

David C. Page

Department of Physics and Astronomy, University of Toronto
Canada
E-mail: page@physics.utoronto.ca

Abstract: We propose a simple form for the superalgebra of M2 and M5-brane probes in arbitrary supersymmetric backgrounds of $11d$ supergravity, extending previous results in the literature. In particular, we identify the topological charges in the algebras and find BPS bounds for the energies. The charges are given by the integral over a brane's spatial worldvolume of a certain closed form built out of the Killing spinors and background fields. The existence of such closed forms for arbitrary supersymmetric backgrounds generalises the existence of calibration forms for special holonomy manifolds.

Keywords: M-Theory, p-branes, Extended Supersymmetry, Supergravity Models.

Contents

1. Introduction

It has been known for some time that the supersymmetry algebras of brane worldvolume theories can contain topological charges which extend the spacetime supersymmetry algebra [1]. As a simple example consider a supermembrane probe in flat eleven dimensional spacetime.

In this case the spacetime superalgebra is the $11d$ super-Poincaré algebra. If we couple a supermembrane probe to this background, the resulting membrane action inherits the symmetries of the background, but with a modification to the supertranslation algebra [1, 2]:

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta} P_M \pm \frac{1}{2}(C\Gamma_{MN})_{\alpha\beta} Z^{MN} \,, \tag{1.1}$$

where

$$Z^{MN} = \int dX^M \wedge dX^N \tag{1.2}$$

and the integration is taken over the spatial worldvolume of the membrane.[1]

Z^{MN} is the integral of a closed form and so the second term in the membrane supertranslation algebra (1.1) depends only on the homology class of the configuration. The existence of such a topological charge in the superalgebra allows massive objects such as branes, which carry the charge, to have supersymmetric ground states.

Subsequent studies have extended this analysis to branes of various types in other fixed supersymmetric backgrounds, see e.g. [3, 4]. In [5] an expression for the membrane and fivebrane superalgebras was given which is valid for a more general class of backgrounds.

[1] Here and in the following, when we write expressions involving forms defined on the background being integrated over the brane worldvolume, a pullback is implied.

Specifically, the analysis of [5] is relevant to backgrounds which have a timelike Killing vector appearing in the supertranslation algebra and which have certain implicit restrictions on the background fields. A further paper [6] studies the M5-brane superalgebra in the presence of non-zero worldvolume fields. The present work proposes a simple and natural generalisation of these results to arbitrary supersymmetric backgrounds and worldvolume fields.

The outline of the paper is as follows. In section 2 we review some recent work on the classification of supersymmetric solutions of $11d$ supergravity [7]. An important step in this classification is the construction of certain forms built out of the Killing spinors which obey differential relations descended from the Killing spinor equations. By simple manipulations we show that these relations are equivalent to the existence of certain closed forms which we will later argue to be the topological charges for branes.

In section 3.1 we review the construction of the spacetime superalgebra associated with a supergravity background [8]. This will help us to fix some useful notations and will suggest a natural construction for the brane superalgebras.

In section 3.2 we consider the superalgebra for a membrane probe and in section 3.3, that of an M5-brane. We make a natural proposal for the form of the superalgebras which agrees with the known examples and makes use of the closed forms constructed in section 2. In section 4, we give an example of the construction applied to the supergravity background sourced by an M5-brane. We conclude with a short discussion.

We note that whilst this paper was being completed [9] appeared which has some overlap with the current work. In particular our expression (3.29) for the BPS bound on the energy/momentum of the fivebrane agrees with [9, equation (4.6)].

2. Killing spinors and closed forms

Recently a great deal of progress has been made in understanding the general structure of supersymmetric solutions of supergravity theories [7] and [10]–[15]. We shall be interested in the case of 11 dimensional supergravity which was investigated by Gauntlett and Pakis [7]. They studied the consequences of having a spinor field which obeys the Killing spinor equations:

$$\tilde{D}_M \epsilon = 0\,, \tag{2.1}$$

where

$$\tilde{D}_M \epsilon \equiv \nabla_M \epsilon + \frac{1}{288}\left[\Gamma_M{}^{NPQR} - 8\delta_M^N \Gamma^{PQR}\right] F_{NPQR}\epsilon \tag{2.2}$$

and F is the four-form field strength of $11d$ supergravity.

In order to study the consequences of (2.1), it is helpful to repackage $\epsilon(x)$ in terms of the following one, two and five-forms:

$$\begin{aligned} K_M &= \bar{\epsilon}\Gamma_M \epsilon \\ \omega_{MN} &= \bar{\epsilon}\Gamma_{MN}\epsilon \\ \Sigma_{MNPQR} &= \bar{\epsilon}\Gamma_{MNPQR}\epsilon\,. \end{aligned} \tag{2.3}$$

It is straightforward to check that the zero, three and four-forms built in a similar way vanish because of the antisymmetry of the relevant Γ matrices. Also $\epsilon(x)$ can be reconstructed (up to a sign) from knowledge of K, ω and Σ. This follows from the completeness of the Γ matrices.

It is not true, however, that an arbitrary set of one, two and five forms are equivalent to a spinor - rather there are algebraic relations between them which follow from Fierz identities. A better way to find these relations uses some algebraic facts about spinors in 10+1 dimensions discussed in [16].

Spinors form a representation of Spin$(1,10)$ and a natural question to ask is what are the possible orbits of a spinor under Spin$(1,10)$. Clearly $K^2 = K^\mu K_\mu$ is a Lorentz scalar and thus is the same along orbits of the group. In fact there are no other independent invariants and Spin$(1,10)$ acts transitively on the level sets of K^2 [16].[2]

Furthermore, the only possibilities for K^2 are $K^2 < 0$ or $K^2 = 0$, i.e. K^μ is either timelike or null. We can choose a convenient normal form for a spinor of either type and deduce that any other such spinor is related by a Lorentz transformation (and possible rescaling if $K^2 < 0$.) Working in this way gives a rather efficient way to define (Lorentz and scaling covariant) identities between the forms K, ω and Σ.

Consider first the case in which $K^2 < 0$. A possible set of projection conditions which define the spinor ϵ is given as follows:

$$\begin{aligned}
\Gamma_{012}\epsilon = \Gamma_{034}\epsilon = \Gamma_{056}\epsilon = \Gamma_{078}\epsilon = \Gamma_{09\natural}\epsilon = \epsilon \\
\Gamma_{013579}\epsilon = \epsilon \,. \qquad\qquad\qquad\qquad
\end{aligned} \tag{2.4}$$

These provide a set of five independent,[3] commuting projections and thus determine a unique spinor up to scale. The scale of the spinor is given by fixing

$$\epsilon^T \epsilon = \Delta \,. \tag{2.5}$$

Using the projection conditions (2.4) we can uniquely determine the forms K, ω and Σ:

$$\begin{aligned}
K &= \Delta e^0 \\
\omega &= \Delta \left(e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6 + e^7 \wedge e^8 + e^9 \wedge e^\natural \right) \\
\Sigma &= \frac{1}{2} \Delta^{-2} K \wedge \omega \wedge \omega + \Delta \mathrm{Re}(\Omega) \,,
\end{aligned} \tag{2.6}$$

where Ω is the holomorphic 5-form:

$$\Omega = \left(e^1 + ie^2\right) \wedge \left(e^3 + ie^4\right) \wedge \left(e^5 + ie^6\right) \wedge \left(e^7 + ie^8\right) \wedge \left(e^9 + ie^\natural\right) . \tag{2.7}$$

Note that K is indeed a timelike vector for this choice of projections and the forms define an SU(5) structure corresponding to the stability group of ϵ. Now, since Spin$(1,10)$ acts

[2]Except that the zero spinor is an obvious fixed point.

[3]Note that one of the six projections given is satisfied automatically as a consequence of the other five and the fact that $\Gamma_{0123456789\natural} \equiv 1$.

JHEP10(2003)005

transitively on the level sets of K^2, we can bring the projection conditions for *any* spinor with $K^2 < 0$ into the standard form given above, by an appropriate choice of vielbein. Thus, a spinor with $K^2 < 0$ is equivalent to a set of forms of the type listed in equation (2.6).

Now consider the case in which K is null. A possible set of projection conditions for the spinor ϵ is given in this case by:

$$\begin{aligned}\Gamma_{01}\epsilon &= -\epsilon \\ \Gamma_{2345}\epsilon &= \Gamma_{2367}\epsilon = \Gamma_{2389}\epsilon = \Gamma_{2468}\epsilon = -\epsilon\,. \end{aligned} \tag{2.8}$$

Note that using the identity $\Gamma_{0123456789\natural} \equiv 1$ we can show that our projectors imply:

$$\Gamma_\natural\epsilon = -\epsilon\,. \tag{2.9}$$

We may use the fact that Lorentz transformations act transitively on the spinors with fixed K^2 to always choose a vielbein such that the projections are of the standard form given in equation (2.8) whenever K is null.

Having fixed the vielbein and the projection conditions we can determine the forms K, ω and Σ uniquely to be:

$$\begin{aligned} K &= \Delta(e^0 + e^1) \equiv \Delta e^+ \\ \omega &= -K \wedge e^\natural \\ \Sigma &= -K \wedge \phi\,, \end{aligned} \tag{2.10}$$

where ϕ is the Cayley four-form:

$$\begin{aligned}\phi = {}& e^2\wedge e^3\wedge e^4\wedge e^5 + e^6\wedge e^7\wedge e^8\wedge e^9 + e^2\wedge e^3\wedge e^6\wedge e^7 - e^2\wedge e^5\wedge e^6\wedge e^9 - \\ & - e^3\wedge e^4\wedge e^7\wedge e^8 + e^2\wedge e^4\wedge e^6\wedge e^8 + e^3\wedge e^5\wedge e^7\wedge e^9 + e^4\wedge e^5\wedge e^8\wedge e^9 + \\ & + e^4\wedge e^5\wedge e^6\wedge e^7 - e^3\wedge e^4\wedge e^6\wedge e^9 + e^2\wedge e^3\wedge e^8\wedge e^9 - e^2\wedge e^5\wedge e^7\wedge e^8 - \\ & - e^2\wedge e^4\wedge e^7\wedge e^9 - e^3\wedge e^5\wedge e^6\wedge e^8\,. \end{aligned} \tag{2.11}$$

Note that K is indeed null. We could set $\Delta \equiv \epsilon^T\epsilon = 1$ by a boost in the $e^+ \equiv e^0 + e^1$ direction but it will be convenient not to do so. The forms K, ω and Σ define a $(\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure [17] corresponding to the stability group of the spinor ϵ [16].

Now, we turn to the differential equations which the forms satisfy as a consequence of $\epsilon(x)$ being a Killing spinor. Expressions for the covariant derivatives of the forms K, ω and Σ are given in [7, equation (2.16)]. We shall be primarily interested in the set of equations for the exterior derivatives of the forms:

$$dK = \frac{2}{3}\iota_\omega F + \frac{1}{3}\iota_\Sigma * F \tag{2.12}$$

$$d\omega = \iota_K F \tag{2.13}$$

$$d\Sigma = \iota_K * F - \omega \wedge F \tag{2.14}$$

and the fact that K^μ is a Killing vector. These equations follow as a result of the Killing spinor equations. Conversely, at least in the case in which K is timelike, this apparently weaker set of equations, supplemented by the algebraic conditions on the forms and the

JHEP10(2003)005

fact that $dF = 0$, is actually strong enough to imply the full set of Killing spinor equations. This surprising fact is highlighted in [7]. It would certainly be interesting to find out if this fact generalises to the case in which K is a null Killing vector.

Now we show that these conditions imply the existence of a certain closed 2-form and a closed 5-form on the manifold. We make repeated use of the following identity. Let X be a vector and α a p-form. Then

$$\mathcal{L}_X \alpha = d(\iota_X \alpha) + \iota_X d\alpha \,, \tag{2.15}$$

where $\mathcal{L}_X$ denotes the Lie derivative in the X direction. Now applying d to equation (2.13) and using $dF = 0$ we find:

$$\mathcal{L}_K F = 0 \,, \tag{2.16}$$

and so K generates a symmetry of the solution. Let A be a 3-form gauge potential for F, $dA = F$. Then we can pick a gauge for A which also preserves the symmetry:

$$\mathcal{L}_K A = 0 \,. \tag{2.17}$$

Now consider the 2-form $\omega + \iota_K A$. This is closed since:

$$d(\omega + \iota_K A) = \iota_K F + \mathcal{L}_K A - \iota_K F = 0 \,. \tag{2.18}$$

We shall see that this closed 2-form is a natural choice for a topological charge in the membrane superalgebra.

Now we construct a closed 5 form to play the same role in the fivebrane superalgebra. Since K is Killing and $\mathcal{L}_K F = 0$, we must also have that:

$$\mathcal{L}_K * F = 0 \,. \tag{2.19}$$

The equations of motion of F state that:

$$d * F + \frac{1}{2} F \wedge F = 0 \,. \tag{2.20}$$

So we can introduce a 6-form gauge potential C for $*F$:

$$dC = *F + \frac{1}{2} A \wedge F \,, \tag{2.21}$$

in such a way that

$$\mathcal{L}_K C = 0 \,. \tag{2.22}$$

Now consider the 5-form $\Sigma + \iota_K C + A \wedge (\omega + \frac{1}{2}\iota_K A)$. This is closed:

$$\begin{aligned} d\left(\Sigma + \iota_K C + A \wedge \left(\omega + \frac{1}{2}\iota_K A\right)\right) &= \iota_K * F - \omega \wedge F + \mathcal{L}_K C - \iota_K \left(*F + \frac{1}{2} A \wedge F\right) + F \wedge \\ &\quad \wedge \left(\omega + \frac{1}{2}\iota_K A\right) - A \wedge \left(\iota_K F + \frac{1}{2}\left(\mathcal{L}_K A - \iota_K F\right)\right) \\ &= 0 \,. \end{aligned} \tag{2.23}$$

This closed 5-form will be the topological charge for an M5-brane.

3. Supersymmetry algebras

3.1 Spacetime supersymmetry algebras

In this section we review the construction [8, 15] of the supersymmetry algebra associated with a solution of eleven dimensional supergravity. As a starting point, consider the super-isometry algebra of $11d$ Minkowski space, i.e. the super-Poincaré algebra. Of particular relevance to the following discussion will be the subalgebra of supertranslations generated by the 32 component Majorana spinor charges Q_α. We have:

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta} P_M \tag{3.1}$$

and

$$[P_M, Q_\alpha] = 0\,, \qquad [P_M, P_N] = 0\,. \tag{3.2}$$

We can rewrite equation (3.1) in an equivalent way by introducing a commuting Majorana spinor parameter ϵ^α and demanding that for arbitrary ϵ^α:

$$\{\epsilon^\alpha Q_\alpha, \epsilon^\beta Q_\beta\} = (\epsilon^T C\Gamma^M \epsilon) P_M\,. \tag{3.3}$$

For Majorana spinors, $\epsilon^T C = \bar{\epsilon}$ and so introducing the vector

$$K^M \equiv \bar{\epsilon}\Gamma^M \epsilon \tag{3.4}$$

this becomes:

$$2(\epsilon Q)^2 = K^M P_M\,. \tag{3.5}$$

Now consider the symmetry algebra of a general solution of eleven dimensional supergravity. Preserved supersymmetries are given by commuting Majorana spinor fields $\epsilon^\alpha(x)$ satisfying the Killing spinor equations (2.2):

$$\tilde{D}_M \epsilon = 0\,. \tag{3.6}$$

For any such Killing spinor $\epsilon^\alpha(x)$ there is a corresponding supercharge ϵQ. Thus the number of linearly independent supercharges is given by the dimension of the space of solutions to the Killing spinor equations. The algebra of these supercharges is given by equation (3.5). Note that K defined in equation (3.4) is now a field also.

We expect $K^M P_M$ to correspond to a bosonic symmetry of the solution. These are given by infinitesimal coordinate transformations which leave the solution invariant. An infinitesimal diffeomorphism is associated with a vector field acting by the Lie derivative. So bosonic symmetries are associated with vector fields $K^M(x)$ which obey Killing's equation

$$\mathcal{L}_K g = 0 \tag{3.7}$$

and also

$$\mathcal{L}_K F = 0\,, \tag{3.8}$$

where F is the four-form field strength of $11d$ supergravity. As we saw in section 2, it is an automatic consequence of the Killing spinor equations that K, constructed as in equation (3.4), obeys these equations. Note that in general, there may be other isometries which are not generated by equation (3.5).

Since the Killing vectors K^M generate infinitesimal coordinate transformations they act on each other by the Lie derivative:

$$\left[K^M P_M, J^N P_N\right] = \left(\mathcal{L}_K J\right)^R P_R\,. \tag{3.9}$$

Under an infinitesimal coordinate transformation which leaves the metric invariant, the vielbein undergoes a Lorentz rotation and so spinors and other objects which transform under change of vielbein are also transformed. Thus the Killing vectors act on spinor fields by the spinorial Lie derivative:[4]

$$\left[K^M P_M, \epsilon^\alpha Q_\alpha\right] = \left(\mathcal{L}_K \epsilon\right)^\beta Q_\beta\,. \tag{3.10}$$

It is a non-trivial consistency check that the super-Jacobi identities are satisfied as an automatic result of this construction [8].

3.2 Supersymmetry algebra for membranes

We now consider the addition of branes. Let's start once again with the super-Poincaré algebra of flat eleven dimensional space. As stated in the introduction, if we couple a supermembrane probe to this background, the resulting membrane action inherits the symmetries of the background, but with a modification to the supertranslation algebra [1]:

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta} P_M \pm \frac{1}{2}(C\Gamma_{MN})_{\alpha\beta} Z^{MN}\,, \tag{3.11}$$

where

$$Z^{MN} = \int dX^M \wedge dX^N \tag{3.12}$$

and the integration is taken over the spatial worldvolume of the membrane. Explicitly, if we introduce coordinates (σ_1, σ_2) on the spatial worldvolume we have:

$$Z^{MN} = \frac{1}{2}\int \epsilon^{ij} \frac{\partial X^M}{\partial \sigma^i}\frac{\partial X^N}{\partial \sigma^j} d\sigma^1 \wedge d\sigma^2\,. \tag{3.13}$$

Similarly, the momentum P_M involves an integration over the spatial worldvolume of a momentum density $p_M(\sigma)$:

$$P_M = \int d^2\sigma\, p_M(\sigma)\,. \tag{3.14}$$

As before, we introduce the constant Majorana spinor parameter ϵ^α and rewrite equation (3.11) as:

$$2(\epsilon Q)^2 = K^M P_M \pm \omega_{MN} Z^{MN}\,, \tag{3.15}$$

where we have also introduced the two-form ω_{MN} defined by:

$$\omega_{MN} = \bar{\epsilon}\Gamma_{MN}\epsilon\,. \tag{3.16}$$

[4]See e.g. [8] for a definition.

We can write this a little more suggestively by taking the (constant) parameters K^M and ω_{MN} inside the integral:

$$2(\epsilon Q)^2 = \int d^2\sigma K^M p_M \pm \int \omega \,. \tag{3.17}$$

A proposal for the generalisation of this formula to membranes in curved $11d$ supergravity backgrounds for which K^M is timelike was presented in [5]. This generalisation was motivated by considerations of kappa-symmetry. In our notation the generalisation of (3.17) to a general curved background (without imposing any restriction on K^M) is:

$$2(\epsilon Q)^2 = \int d^2\sigma K^M p_M \pm \int (\omega + \iota_K A) \,. \tag{3.18}$$

In this expression, A is a three-form potential for the four-form field strength F.[5] Note that K and ω are no longer constant, but are the fields built from the Killing spinors according to equations (3.4) and (3.16). Also note, that a particular gauge choice for A has been taken so that

$$\mathcal{L}_K A = 0 \,. \tag{3.19}$$

This is possible to do since $\mathcal{L}_K F = 0$ and just corresponds to a choice of gauge potential A which preserves the symmetry generated by K. In order for our proposal to make sense, we require that the second term in (3.18) be topological, i.e.

$$d(\omega + \iota_K A) = 0 \,. \tag{3.20}$$

As we saw in section 2, this equation is a consequence of the Killing spinor equations and the definition of ω.

The supersymmetry algebra (3.18) leads to a BPS type bound on the energy/momentum of the M2-brane, since $(\epsilon Q)^2 \geq 0$. We find:

$$\int d^2\sigma K^M p_M \geq \mp \int (\omega + \iota_K A) \,, \tag{3.21}$$

where the term on the RHS is topological. This bound was found for the case of timelike K^M in [5].

3.3 Supersymmetry algebra for five-branes

In flat space the supertranslation algebra for a five-brane probe is:

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta} P_M \pm \frac{1}{5!}(C\Gamma_{MNPQR})_{\alpha\beta} Z^{MNPQR} \,, \tag{3.22}$$

where

$$Z^{MNPQR} = \int dX^M \wedge dX^N \wedge dX^P \wedge dX^Q \wedge dX^R \tag{3.23}$$

and the integration is taken over the spatial worldvolume of the five-brane.

[5] The combination of $K^M P_M \pm \iota_K A$ is very natural since the brane is electrically charged under A and this generalises the replacement of p_μ with $p_\mu + eA_\mu$ for a charged particle in an electromagnetic field.

Written in terms of the parameter ϵ this becomes:

$$2(\epsilon Q)^2 = K^M P_M \pm \Sigma_{MNPQR} Z^{MNPQR}, \tag{3.24}$$

where we have also introduced the five-form Σ_{MNPQR} defined by:

$$\Sigma_{MNPQR} = \bar{\epsilon}\Gamma_{MNPQR}\epsilon\,. \tag{3.25}$$

To extend this to a curved spacetime background we let $\epsilon(x)$ be a Killing spinor field and following the membrane case we might guess that the algebra becomes:

$$2(\epsilon Q)^2 = \int d^5\sigma K^M p_M(\sigma) \pm \int (\iota_K C + \Sigma)\,, \tag{3.26}$$

where C is the gauge potential for F. Actually, as we saw in section 2, this is not closed and so the correct answer is:

$$2(\epsilon Q)^2 = \int d^5\sigma K^M p_M(\sigma) \pm \int \left(\iota_K C + \Sigma + A \wedge \left(\omega + \frac{1}{2}\iota_K A\right)\right). \tag{3.27}$$

It is also interesting to consider the supersymmetry algebra with non-zero worldvolume gauge field B. We can construct a closed five-form using the closed two-form $\omega + \iota_K A$ and the closed three-form dB. With such a term, the supersymmetry algebra becomes:

$$2(\epsilon Q)^2 = \int d^5\sigma K^M p_M(\sigma) \pm \int \left(\iota_K C + \Sigma + (A + dB) \wedge (\omega + \iota_K A) - \frac{1}{2} A \wedge \iota_K A\right). \tag{3.28}$$

This equation agrees with and generalises results in [3, 6], for the M5-brane supersymmetry algebra with non-zero worldvolume fields.[6]

Once, again the supersymmetry algebra, (3.28) leads to a BPS bound on the energy/momentum:

$$\int d^5\sigma K^M p_M(\sigma) \geq \mp \int \left(\iota_K C + \Sigma + (A + dB) \wedge (\omega + \iota_K A) - \frac{1}{2} A \wedge \iota_K A\right). \tag{3.29}$$

4. An example: M5-brane background

We now present an example to illustrate the general approach for constructing the superalgebras of brane probes and to point out a subtlety in our expressions for the topological charges. First, we need to choose a supersymmetric background in which to work. We choose to consider the supergravity background corresponding to a collection of coincident M5-branes. We will also need to choose a specific supersymmetry of the background and we will choose one such that $K^2 = 0$.

[6] Note that the coefficient of the dB term in the previous equation is not fixed by the requirement that the five-form be closed, since we are adding together two five-forms which are individually closed. We can fix the normalisation by reference to the flat space analysis of [6].

JHEP10(2003)005

The metric and 7-form are given by

$$\begin{aligned} ds^2 &= H^{-1/3}\left(-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2+(dx^4)^2+(dx^5)^2\right)+\\ &\quad + H^{2/3}\Big((dx^6)^2+(dx^7)^2+(dx^8)^2+(dx^9)^2+(dx^\natural)^2\Big) && (4.1)\\ *F &= -dH^{-1}\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge dx^4\wedge dx^5\,, && (4.2)\end{aligned}$$

where H is a harmonic function which depends on the radial distance, r, from the brane where

$$r^2=(x^6)^2+(x^7)^2+(x^8)^2+(x^9)^2+(x^\natural)^2\,.$$

We also have:

$$F=\frac{1}{r}\,\frac{\partial H}{\partial r}\,\frac{1}{4!}\,\epsilon_{ijklm}x^i dx^j\wedge dx^k\wedge dx^l\wedge dx^m\,, \tag{4.3}$$

where $i,j,\ldots$ run over the indices $\{6,7,8,9,\natural\}$.

The background has 16 Killing spinors $\epsilon=H^{-1/12}\epsilon_0$ which are constructed from constant spinors ϵ_0 satisfying:

$$\Gamma_{012345}\epsilon_0=\epsilon_0\,. \tag{4.4}$$

Note that if we normalise ϵ_0 such that $\epsilon_0^T\epsilon_0=1$ then $\epsilon^T\epsilon=H^{-1/6}$. One can make the projections of equation (2.8) on the Killing spinors of the background:

$$\begin{aligned}\Gamma_{01}\epsilon&=-\epsilon\\ \Gamma_{2345}\epsilon=\Gamma_{2367}\epsilon&=\Gamma_{2468}\epsilon=\Gamma_{2389}\epsilon=-\epsilon\,. \end{aligned}\tag{4.5}$$

These projections are consistent with the projection condition for the the background M5-brane: $\Gamma_{012345}\epsilon=\epsilon$.

As discussed in section 2, the above conditions define a null Killing vector, K. In this background K, ω and Σ are given explicitly by

$$\begin{aligned} K &= -H^{-1/3}(dx^0+dx^1) && (4.6)\\ \omega &= -K\wedge e^\natural=(dx^0+dx^1)\wedge dx^\natural && (4.7)\\ \Sigma &= -K\wedge\phi\,, && (4.8)\end{aligned}$$

where ϕ is the Cayley 4-form given by

$$\begin{aligned}\phi = {}& H^{-2/3}dx^2\wedge dx^3\wedge dx^4\wedge dx^5+H^{4/3}dx^6\wedge dx^7\wedge dx^8\wedge dx^9\times\\ &\times H^{1/3}\Big[dx^2\wedge dx^3\wedge dx^6\wedge dx^7-dx^3\wedge dx^4\wedge dx^7\wedge dx^8+dx^2\wedge dx^4\wedge dx^6\wedge dx^8+\\ &\quad+dx^3\wedge dx^5\wedge dx^7\wedge dx^9-dx^2\wedge dx^5\wedge dx^6\wedge dx^9+dx^4\wedge dx^5\wedge dx^8\wedge dx^9+\\ &\quad+dx^4\wedge dx^5\wedge dx^6\wedge dx^7-dx^3\wedge dx^4\wedge dx^6\wedge dx^9+dx^2\wedge dx^3\wedge dx^8\wedge dx^9-\\ &\quad-dx^2\wedge dx^5\wedge dx^7\wedge dx^8-dx^2\wedge dx^4\wedge dx^7\wedge dx^9-dx^3\wedge dx^5\wedge dx^6\wedge dx^8\Big]\,.\end{aligned}$$

We now check that the differential equations for K,ω and Σ, given in eqs. (2.12)–(2.14) are satisfied. Given the form of F it is clear that $\iota_K F=\iota_\omega F=0$. Clearly from eq. (4.7) we have

$$d\omega=0=\iota_K F \tag{4.9}$$

JHEP10(2003)005

and thus eq. (2.13) for ω is satisfied. We now consider the differential equation (2.12) for K. From the explicit form of K in eq. (4.6) we have

$$dK = -\frac{1}{3}H^{-4/3}(dx^0 + dx^1) \wedge dH\,. \tag{4.10}$$

Since $\iota_\omega F = 0$ we simply have to show that this is equal to $\frac{1}{3}\iota_\Sigma * F$. From the form of $*F$ it is clear that the only term in Σ that makes a non-zero contribution to $\iota_\Sigma * F$ is $-K \wedge H^{-2/3}dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5$, which comes from the first term in the Cayley 4-form. Therefore we find

$$\iota_\Sigma * F = H^{2/3}(dx^0 + dx^1) \wedge dH^{-1} = -H^{-4/3}(dx^0 + dx^1) \wedge dH = 3dK \tag{4.11}$$

and so eq. (2.12) is satisfied. Finally we verify that the differential equation for Σ, eq. (2.14) holds. Now

$$d\Sigma = dH \wedge (dx^0 + dx^1) \wedge \left(-H^{-2}\; dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 + dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9\right). \tag{4.12}$$

We can also calculate

$$\iota_K * F = -H^{-2}dH \wedge (dx^0 + dx^1) \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \tag{4.13}$$

which clearly agrees with the first term in $d\Sigma$. Also

$$\begin{aligned} \omega \wedge F &= \frac{\partial H}{\partial r}(dx^0 + dx^1) \wedge \frac{x^\natural dx^\natural}{r} \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9 \\ &= \frac{\partial H}{\partial r}(dx^0 + dx^1) \wedge dr \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9 \end{aligned} \tag{4.14}$$

so from eqs. (4.12)–(4.14) we see that

$$d\Sigma = \iota_K * F - \omega \wedge F \tag{4.15}$$

as required.

We now find the closed two and five forms appearing in the membrane and fivebrane superalgebras. We can choose a gauge for A with $\mathcal{L}_K A = 0$, such that $\iota_K A = 0$ and then the two form is just ω. The non-zero terms in the five form are then

$$\iota_K C + \Sigma + (A + dB) \wedge \omega\,. \tag{4.16}$$

We can pick C so that

$$\iota_K C = -(H^{-1} - 1)(dx^0 + dx^1) \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5\,. \tag{4.17}$$

The remaining subtlety is in how to define $A \wedge \omega$ since A is a magnetic potential for the background solution and not globally well-defined. The natural solution in this case is to define the integral of $A \wedge \omega$ over the 5d spatial worldvolume of the brane via an integral of $F \wedge \omega$ over a 6d surface whose boundary is the 5d spatial worldvolume. From the expression for $F \wedge \omega$ above we see that this can be simply integrated to give back a 5d integral of

$$-(H - 1) \wedge (dx^0 + dx^1) \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9\,. \tag{4.18}$$

Putting everything together we find that the expression for the five form is just

$$(dx^0 + dx^1) \wedge \phi_f + dB \wedge \omega \,, \tag{4.19}$$

where ϕ_f is just the flat space Cayley four-form. This expression is manifestly closed. We see that for a choice of supersymmetry which is preserved by the background brane, the supersymmetry algebra for membrane and fivebrane probes is unaltered from flat space.

5. Discussion

We have presented expressions for the supersymmetry algebras of membranes and M5-branes in arbitrary supersymmetric backgrounds of eleven dimensional supergravity. In particular, we have shown how supersymmetry ensures the existence of closed two and five forms which appear as topological charges in the algebras. It should be straightforward to apply the same ideas to other supergravity theories in different dimensions, giving a simple derivation of the supersymmetry algebras for the branes in these theories.

One motivation for our work was to understand better the dynamics of branes which preserve supersymmetries related to null Killing spinors. Examples of such branes are giant gravitons [18]–[21], null intersecting branes [17, 22, 23], supertubes/M-ribbons [24]–[27].The analysis of the supersymmetry algebras and related BPS bounds which we present here is a useful step towards this understanding, but there is certainly more to be done here. It would be very interesting to find a way of describing the most general brane configuration which preserves the supersymmetry associated with a particular Killing spinor.

It would also be interesting to extend our analysis to cases in which there are several Killing spinors and to understand better the structure of the supersymmetry algebra in those cases. A first step for doing this would be to classify normal forms for projection conditions preserving different numbers of supersymmetries in order to find the algebraic structure of the associated forms.

Acknowledgments

We would like to thank Jerome Gauntlett for helpful comments on an early draft of the paper. D.C.P. thanks Kazuo Hosomichi, Rob Myers and Amanda Peet for discussions and is grateful to the Perimeter Institute for hospitality during the completion of this paper. D.C.P. is supported by Ontario PREA. E.J.H. is funded by the University of Adelaide and the Overseas Research Students Awards Scheme.

References

[1] J.A. de Azcarraga, J.P. Gauntlett, J.M. Izquierdo and P.K. Townsend, *Topological extensions of the supersymmetry algebra for extended objects*, *Phys. Rev. Lett.* **63** (1989) 2443.

[2] P.K. Townsend, *M-theory from its superalgebra*, hep-th/9712004.

[3] D.P. Sorokin and P.K. Townsend, *M-theory superalgebra from the M-5-brane*, *Phys. Lett.* **B 412** (1997) 265 [hep-th/9708003].

[4] T. Sato, *Various supersymmetric brane configurations from superalgebras in many types of M-brane backgrounds*, *Nucl. Phys.* **B 548** (1999) 231 [hep-th/9812014].

[5] J. Gutowski, G. Papadopoulos and P.K. Townsend, *Supersymmetry and generalized calibrations*, *Phys. Rev.* **D 60** (1999) 106006 [hep-th/9905156].

[6] O. Barwald, N.D. Lambert and P.C. West, *A calibration bound for the M-theory fivebrane*, *Phys. Lett.* **B 463** (1999) 33 [hep-th/9907170].

[7] J.P. Gauntlett and S. Pakis, *The geometry of $D = 11$ killing spinors*, *J. High Energy Phys.* **04** (2003) 039 [hep-th/0212008].

[8] J.M. Figueroa-O'Farrill, *On the supersymmetries of anti de Sitter vacua*, *Class. and Quant. Grav.* **16** (1999) 2043 [hep-th/9902066].

[9] D. Martelli and J. Sparks, *G-structures, fluxes and calibrations in M-theory*, hep-th/0306225.

[10] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, *Class. and Quant. Grav.* **20** (2003) 4587 [hep-th/0209114].

[11] J.P. Gauntlett and J.B. Gutowski, *All supersymmetric solutions of minimal gauged supergravity in five dimensions*, hep-th/0304064.

[12] J.P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with intrinsic torsion*, hep-th/0302158.

[13] M.J. Duff and J.T. Liu, *Hidden spacetime symmetries and generalized holonomy in M-theory*, hep-th/0303140.

[14] C. Hull, *Holonomy and symmetry in M-theory*, hep-th/0305039.

[15] G. Papadopoulos and D. Tsimpis, *The holonomy of the supercovariant connection and Killing spinors*, *J. High Energy Phys.* **07** (2003) 018 [hep-th/0306117].

[16] R.L. Bryant, *Pseudo-Riemannian metrics with parallel spinor fields and non-vanishing Ricci tensor*, math.DG/0004073.

[17] B.S. Acharya, J.M. Figueroa-O'Farrill, B. Spence and S. Stanciu, *Planes, branes and automorphisms. ii: branes in motion*, *J. High Energy Phys.* **07** (1998) 005 [hep-th/9805176].

[18] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *J. High Energy Phys.* **06** (2000) 008 [hep-th/0003075].

[19] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and goliath*, *J. High Energy Phys.* **08** (2000) 040 [hep-th/0008015].

[20] A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *J. High Energy Phys.* **08** (2000) 051 [hep-th/0008016].

[21] A. Mikhailov, *Giant gravitons from holomorphic surfaces*, *J. High Energy Phys.* **11** (2000) 027 [hep-th/0010206].

[22] C. Bachas and C. Hull, *Null brane intersections*, *J. High Energy Phys.* **12** (2002) 035 [hep-th/0210269].

[23] R.C. Myers and D.J. Winters, *From D-anti-D pairs to branes in motion*, *J. High Energy Phys.* **12** (2002) 061 [hep-th/0211042].

JHEP10(2003)005

[24] D. Mateos and P.K. Townsend, *Supertubes*, *Phys. Rev. Lett.* **87** (2001) 011602 [hep-th/0103030].

[25] R. Emparan, D. Mateos and P.K. Townsend, *Supergravity supertubes*, *J. High Energy Phys.* **07** (2001) 011 [hep-th/0106012].

[26] D. Mateos, S. Ng and P.K. Townsend, *Supercurves*, *Phys. Lett.* **B 538** (2002) 366 [hep-th/0204062].

[27] Y. Hyakutake and N. Ohta, *Supertubes and supercurves from M-ribbons*, *Phys. Lett.* **B 539** (2002) 153 [hep-th/0204161].